

Joint secrecy over the K -Transmitter Multiple Access Channel

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Abstract—This paper studies the problem of secure communication over a K -transmitter multiple access channel in the presence of an external eavesdropper, subject to a joint secrecy constraint (i.e., information leakage rate from the collection of K messages to an eavesdropper is made vanishing). As a result, we establish the joint secrecy achievable rate region. To this end, our results build upon two techniques in addition to the standard information-theoretic methods. The first is a generalization of Chia-El Gamal's lemma on entropy bound for a set of codewords given partial information. The second is to utilize a compact representation of a list of sets that, together with properties of mutual information, leads to an efficient Fourier-Motzkin elimination. These two approaches could also be of independent interests in other contexts.

I. INTRODUCTION

For the problem of reliably communicating independent messages over a multiple access channel (MAC), Ahlswede [1] first studied the 2-transmitter and 3-transmitter cases and determined the respective capacity regions; whilst Liao [2] considered the general K -transmitter MAC and fully characterized its capacity region.

Inspired by the pioneering works of Wyner [3] and Csiszár and Körner [4] that studied the information theoretic secrecy of a point-to-point communication in the presence of an external eavesdropper, a MAC with an external eavesdropper was first introduced in [5]. In particular, [5] focused on a degraded Gaussian MAC with K -transmitters and established achievable rate regions subject to a pre-specified secrecy measure; while a discrete memoryless 2-transmitter MAC with an external eavesdropper was considered in [6]. Further works on MAC with an external eavesdropper include but not limited to [7]–[10]. However, its secrecy capacity region, even for the 2-transmitter case, still remains an open problem.

In this paper, we consider the secure communication over a K -transmitter MAC subject to the joint secrecy constraint (i.e., information leakage rate from the collection of K messages to an eavesdropper is made vanishing), the channel model of which is shown in Fig. 1. As a general result, we establish a joint secrecy achievable rate region.

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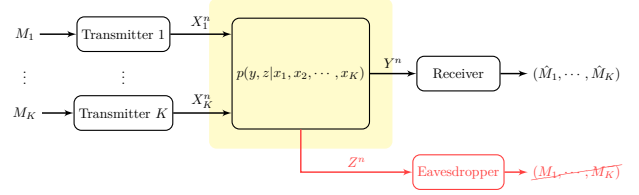


Fig. 1: K -transmitter DM-MAC with an eavesdropper.

The rest of the paper is organized as follows. Section II introduces the system model and presents necessary definitions; Section III gives the general conditional entropy bound. The bound plays an important role in establishing the joint secrecy rate region that is given in Section III. To enhance the flow of the paper, some details for the Fourier-Motzkin elimination are relegated to Appendix A.

II. PRELIMINARIES

Consider a discrete memoryless MAC (DM-MAC) with K transmitters, one legitimate receiver, and one passive eavesdropper, which is defined by $p(y, z | x_1, x_2, \dots, x_K)$. The transmitter i , aims to send message m_i , to the legitimate receiver, where $i \in \mathcal{K} = \{1, 2, \dots, K\}$. Suppose that x_i^n is the channel input at transmitter i , and the channel outputs at the legitimate receiver and eavesdropper are y^n and z^n , respectively. By the *discrete memoryless* nature of the channel (without any feedback), we have

$$p(y^n, z^n | x_1^n, \dots, x_K^n) = \prod_{i=1}^n p(y_i, z_i | x_{1,i}, \dots, x_{K,i}). \quad (1)$$

A $(2^{nR_1}, 2^{nR_2}, \dots, 2^{nR_K}, n)$ secrecy code \mathcal{C}_n for the DM-MAC consists of

- K message sets $\mathcal{M}_1, \mathcal{M}_2, \dots, \mathcal{M}_K$, where $m_i \in \mathcal{M}_i = [1 : 2^{nR_i}]$ for $i \in \mathcal{K}$;
- K encoders each assigning a codeword x_i^n to message m_i for $i \in \mathcal{K}$; and
- One decoder at the legitimate receiver that declares an estimate of (m_1, m_2, \dots, m_K) say $(\hat{m}_1, \hat{m}_2, \dots, \hat{m}_K)$ or an error to the received sequence y^n .

In this paper, for a list of random variables W_j for $j \in \mathcal{K}$, and a fixed $\mathcal{J} \subseteq \mathcal{K}$, we denote $W_{\mathcal{J}} = \{W_i | i \in \mathcal{J}\}$. Assume that the messages M_K are uniformly distributed

over their corresponding message sets. Therefore, we have $R_i = \frac{1}{n}H(M_i)$, for $i \in \mathcal{K}$. Denote the *average probability of decoding error* at the legitimate receiver as $P_e^n(\mathcal{C}_n) = \Pr \left\{ \bigcup_{i \in \mathcal{K}} \{M_i \neq \hat{M}_i\} | \mathcal{C}_n \right\}$. Define the *joint information leakage rate* to the eavesdropper by $R_{L,\mathcal{K}}(\mathcal{C}_n) = \frac{1}{n}I(M_{\mathcal{K}}; Z^n | \mathcal{C}_n)$. The rate pair (R_1, R_2, \dots, R_K) is said to be *achievable under the joint secrecy constraint*, if there exists a sequence of $(2^{nR_1}, 2^{nR_2}, \dots, 2^{nR_K}, n)$ codes $\{\mathcal{C}_n\}$ such that

$$P_e^n(\mathcal{C}_n) \leq \epsilon_n, \quad (2)$$

$$R_{L,\mathcal{K}}(\mathcal{C}_n) \leq \tau_n, \quad (3)$$

$$\lim_{n \rightarrow \infty} \epsilon_n = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} \tau_n = 0. \quad (4)$$

Recall that $\mathcal{K} = \{1, 2, \dots, K\}$. We have the following definition and lemmas.

Definition 1. The *indicator vector* of a subset \mathcal{T} of set \mathcal{K} , denoted by $1_{\mathcal{T}}$, is a $1 \times K$ vector, with its i -th element equal to 1 if $i \in \mathcal{T}$ and 0 otherwise, for $1 \leq i \leq K$.

For instance, for $K = 5$, $\mathcal{K} = \{1, 2, 3, 4, 5\}$ and $\mathcal{T} = \{1, 3, 5\}$, we have $1_{\mathcal{T}} = [1 \ 0 \ 1 \ 0 \ 1]$, and $1_{\emptyset} = [0 \ 0 \ 0 \ 0 \ 0]$.

Let $\{\mathcal{T}_i | 1 \leq i \leq t\}$ be a list of t subsets of \mathcal{K} .

Definition 2. The *presence vector* of $\{\mathcal{T}_i | 1 \leq i \leq t\}$ is defined to be $\mathbf{t}_{\#} = \sum_{i=1}^t 1_{\mathcal{T}_i}$, which counts the number of presences of each element of \mathcal{K} over $\{\mathcal{T}_i | 1 \leq i \leq t\}$.

Definition 3. A *compact form* of the element rearrangement for $\{\mathcal{T}_i | 1 \leq i \leq t\}$ is defined to be $\{\mathcal{T}_{t,i}^* | 1 \leq i \leq t\}$, where $\mathcal{T}_{t,i}^*$ contains the elements that present at least i times from all these t subsets, i.e.,

$$\mathcal{T}_{t,i}^* = \bigcup_{\{j_1, \dots, j_i\} \subseteq [1:t]} \left(\bigcap_{k=1}^i \mathcal{T}_{j_k} \right).$$

Clearly, $\mathcal{T}_{t,i}^* \subseteq \mathcal{T}_{t,i-1}^* \subseteq \dots \subseteq \mathcal{T}_{t,1}^*$. And, $\mathcal{T}_{t,i}^* = \emptyset$ for $i > t_{\max}$, where t_{\max} is the largest element of $\mathbf{t}_{\#}$. So $\{\mathcal{T}_{t,i}^* | 1 \leq i \leq t_{\max}\}$ is the compact form without the empty sets.

Lemma 4. $\mathbf{t}_{\#} = \sum_{i=1}^t 1_{\mathcal{T}_i} = \sum_{i=1}^t 1_{\mathcal{T}_{t,i}^*} = \sum_{i=1}^{t_{\max}} 1_{\mathcal{T}_{t,i}^*}$.

For instance, for $K = 3$, $\mathcal{K} = \{1, 2, 3\}$ and $\{\mathcal{T}_i | 1 \leq i \leq 3\}$ with $\mathcal{T}_1 = \{1\}$, $\mathcal{T}_2 = \{1, 2\}$, $\mathcal{T}_3 = \{2, 3\}$. We have $\mathbf{t}_{\#} = (2, 2, 1)$, $t_{\max} = 2$, $\mathcal{T}_{3,1}^* = \{1, 2, 3\}$, $\mathcal{T}_{3,2}^* = \{1, 2\}$, $\mathcal{T}_{3,3}^* = \emptyset$.

Lemma 5. Given two lists of sets $\{\mathcal{T}_{1i} | 1 \leq i \leq t_1\}$ and $\{\mathcal{T}_{2i} | 1 \leq i \leq t_2\}$, elements of which are t_1 and t_2 subsets of set \mathcal{K} , respectively, if they share the same presence sequence $\mathbf{t}_{\#}$, then they also share the same compact form of the element rearrangement without empty sets.

III. CONDITIONAL ENTROPY BOUND

In this section, we give the conditional entropy bound, which is a generalization of Chia-El Gamal's lemma [11, Lemma 1] on entropy bound for a set of conditionally independent codewords given partial information.

Lemma 6. Let $R_{v,i} \geq 0$ for $i \in \mathcal{K}$, $\epsilon > 0$, and $(Q, V_1, \dots, V_K, Z) \sim p(q) \cdot \prod_{i \in \mathcal{K}} p(v_i | q) \cdot p(z | v_1, \dots, v_K)$. Let Q^n be a random sequence and each $q^n = (q(1), \dots, q(n))$ distributed according to $\prod_{t=1}^n p(q(t))$. For $i \in \mathcal{K}$, let $V_i^n(l_i)$, $l_i \in [1 : 2^{nR_{v,i}}]$, be a set of random sequences that are conditionally independent given Q^n and each $v_i^n = (v_i(1), \dots, v_i(n))$ distributed according to $\prod_{t=1}^n p(v_i(t) | q(t))$, and let \mathcal{C} be the codebook of $(Q^n, V_1^n(1), \dots, V_K^n(2^{nR_{v,K}}))$. Let L_i be the random index of V_i^n , for $i \in \mathcal{K}$, with an arbitrary probability mass function. Then, if $\Pr\{(Q^n, V_1^n(L_1), \dots, V_K^n(L_K), Z^n) \in \mathcal{T}_{\epsilon}^n(Q, V_1, \dots, V_K, Z)\} \rightarrow 1$ as $n \rightarrow \infty$ and

$$\sum_{j \in \mathcal{J}} R_{v,j} \geq I(V_{\mathcal{J}}; Z | Q), \quad \forall \mathcal{J} \subseteq \mathcal{K} \quad (5)$$

there exists a $\delta_n(\epsilon) \rightarrow 0$ as $\epsilon \rightarrow 0$ and $n \rightarrow \infty$, such that for n sufficiently large, $H(L_1, \dots, L_K | Z^n, Q^n, \mathcal{C}) \leq n \left[\sum_{j \in \mathcal{K}} R_{v,j} - I(V_{\mathcal{K}}; Z | Q) \right] + n\delta_n(\epsilon)$.

Proof: Given z^n , let us define \mathcal{L} as the set of indices (l_1, l_2, \dots, l_K) such that

$$(q^n, v_1^n(l_1), \dots, v_K^n(l_K), z^n) \in \mathcal{T}_{\epsilon}^n(Q, V_1, \dots, V_K, Z).$$

First we show that the expected size of this list, over all randomly generated codebooks, is upper bounded by

$$\mathbb{E}(|\mathcal{L}|) \leq 1 + \sum_{i=1}^{2^K-1} 2^{n[I_i + \delta(\epsilon)]}, \quad (6)$$

where $I_i = \sum_{j \in \mathcal{J}_i} R_{v,j} - I(V_{\mathcal{J}_i}; Z | V_{\mathcal{J}_i^c}, Q)$. Here, $\{\mathcal{J}_i | i \in [1 : 2^K - 1]\}$ are the $2^K - 1$ non-empty subsets of \mathcal{K} , and $\mathcal{J}_i^c = \mathcal{K} \setminus \mathcal{J}_i$ for $i \in [1 : 2^K - 1]$. Note that

$$\begin{aligned} \mathbb{E}(|\mathcal{L}|) &= \Pr\{(L_1, \dots, L_K) \in \mathcal{L}\} \\ &+ \sum_{(l_1, \dots, l_K) \neq (L_1, \dots, L_K)} \Pr\{(l_1, \dots, l_K) \in \mathcal{L}\}, \end{aligned}$$

where (L_1, \dots, L_K) are the true indices chosen by the sources. Since $\Pr\{(q^n, v_1^n(L_1), \dots, v_K^n(L_K), z^n) \in \mathcal{T}_{\epsilon}^n(Q, V_1, \dots, V_K, Z)\} \rightarrow 1$ as $n \rightarrow \infty$, the 1st term tends to 1 as $n \rightarrow \infty$. As for the 2nd term, we can distinguish $(2^K - 1)$ cases according to the values of (l_1, l_2, \dots, l_K) . More specifically, for each \mathcal{J}_i , $i \in [1 : 2^K - 1]$, we consider the following case:

- $l_j \neq L_j$ for $j \in \mathcal{J}_i$, and $l_j = L_j$ for $j \in \mathcal{J}_i^c$.

In this case, in total there are at most $2^{\sum_{j \in \mathcal{J}_i} R_{v,j}}$ possible (l_1, \dots, l_K) . By the joint typicality lemma, we can upper bound $\Pr\{(l_1, \dots, l_K) \in \mathcal{L}\}$ by

$$2^{-nI(V_{\mathcal{J}_i}; V_{\mathcal{J}_i^c}, Z | Q) + n\delta(\epsilon)} \stackrel{(a)}{=} 2^{-nI(V_{\mathcal{J}_i}; Z | V_{\mathcal{J}_i^c}, Q) + n\delta(\epsilon)},$$

where (a) is due to the fact that $V_{\mathcal{J}_i}$ and $V_{\mathcal{J}_i^c}$ are conditionally independent given Q .

Therefore, in this case, there are at most $2^{n[I_i + \delta(\epsilon)]}$ number of (l_1, l_2, \dots, l_K) falling in the list \mathcal{L} .

Summing up all the numbers of (l_1, l_2, \dots, l_K) falling in the list \mathcal{L} over all these $(2^K - 1)$ cases, we prove (6).

Furthermore, define the indicator variable $E = 1$ if $(L_1, \dots, L_K) \in \mathcal{L}$, and $E = 0$ otherwise. We have

$$\begin{aligned} H(L_1, \dots, L_K | Z^n, Q^n, \mathcal{C}) &\leq H(L_1, \dots, L_K, E | Z^n, Q^n, \mathcal{C}) \\ &\leq H(E) + H(L_1, \dots, L_K | Z^n, Q^n, E, \mathcal{C}) \\ &\stackrel{(b)}{\leq} 1 + H(L_1, \dots, L_K | Z^n, Q^n, E = 1, \mathcal{C}) \\ &\quad + \Pr\{E = 0\} H(L_1, \dots, L_K | \mathcal{C}), \end{aligned}$$

where (b) follows from the fact that $H(E) \leq 1$ since E is a binary random variable; $\Pr\{E = 1\} \leq 1$ and conditioning does not increase the entropy.

Since $\Pr\{(q^n, v_1^n(L_1), \dots, v_K^n(L_K), z^n) \in \mathcal{T}_\epsilon^n(Q, V_1, \dots, V_K, Z)\} \rightarrow 1$ as $n \rightarrow \infty$, then $\Pr\{E = 0\} = \Pr\{(L_1, L_2, \dots, L_K) \notin \mathcal{L}\}$ can be made arbitrarily small as $n \rightarrow \infty$. Next,

$$\begin{aligned} &H(L_1, \dots, L_K | Z^n, Q^n, E = 1, \mathcal{C}) \\ &\stackrel{(c)}{=} H(L_1, \dots, L_K | Z^n, Q^n, E = 1, \mathcal{C}, \mathcal{L}, |\mathcal{L}|) \\ &\leq H(L_1, \dots, L_K | E = 1, \mathcal{L}, |\mathcal{L}|) \\ &= \sum_{l \in \text{supp}(|\mathcal{L}|)} \Pr\{|\mathcal{L}| = l\} H(L_1, \dots, L_K | E = 1, \mathcal{L}, |\mathcal{L}| = l) \\ &\stackrel{(d)}{\leq} \sum_{l \in \text{supp}(|\mathcal{L}|)} \Pr\{|\mathcal{L}| = l\} \log_2(l) \\ &= \mathbb{E}(\log_2(|\mathcal{L}|)) \stackrel{(e)}{\leq} \log_2(\mathbb{E}(|\mathcal{L}|)) \\ &\stackrel{(f)}{\leq} n \max\{0, \max_{i \in [1:2^K-1]} I_i\} + K + n\delta(\epsilon) \\ &\stackrel{(g)}{\leq} n \left[\sum_{j \in \mathcal{K}} R_{v,j} - I(V_{\mathcal{K}}; Z | Q) \right] + K + n\delta(\epsilon), \end{aligned}$$

where (c) follows from the fact that \mathcal{L} and $|\mathcal{L}|$ are functions of the output Z^n , given \mathcal{C} and Q^n ; (d) is due to the fact that, knowing $E = 1$, the sent indices (L_1, L_2, \dots, L_K) belong to the list \mathcal{L} and the uncertainty is upper bounded by the log cardinality of the list; (e) is by Jensen's inequality; (f) is by (6) along with an application of the log-sum-exp inequality: $\log_2\left(\sum_{x \in \mathcal{X}} 2^x\right) \leq \max_{x \in \mathcal{X}} x + \log_2(|\mathcal{X}|)$; and (g) follows if the rates satisfies (5), i.e.: $\sum_{j \in \mathcal{J}} R_{v,j} \geq I(V_{\mathcal{J}}; Z)$, $\forall \mathcal{J} \subseteq \mathcal{K}$. This, along with previous remarks yields the desired inequality (by defining $\delta_n(\epsilon)$ to be the arbitrary small term $\mathcal{O}(\epsilon) + (K + 1)/n$). ■

IV. ACHIEVABLE RATE REGION

In this section, we give an achievable joint secrecy rate region of the K -transmitter DM-MAC with an external eavesdropper. This result recovers the joint secrecy result for $K = 2$ in [12, Theorem 2], which improves [6, (8)] with channel prefixing as demonstrated in [12].

Theorem 7. *An achievable joint secrecy rate region of the K -transmitter DM-MAC with an external eavesdropper is given by the union of non-negative rate pairs (R_1, R_2, \dots, R_K) that are defined by the followings:*

$$\sum_{j \in \mathcal{J}} R_j \leq I(V_{\mathcal{J}}; Y | V_{\mathcal{J}^c}, Q) - I(V_{\mathcal{J}}; Z | Q), \quad \forall \mathcal{J} \subseteq \mathcal{K}$$

where the union is over input probability distributions that factor as $p(q) \prod_{i \in \mathcal{K}} p(v_i | q) p(x_i | v_i)$.

Proof: Fix $p(q)$ and $p(v_i | q), p(x_i | v_i)$ for $i \in \mathcal{K}$. Generate a random sequence q^n , where $p(q^n) = \prod_{t=1}^n p(q(t))$ with each entry chosen as i.i.d. $p(q)$. The sequence q^n is given to every node in the system.

Codebook generation: For $i \in \mathcal{K}$, to construct codebook \mathcal{C}_i , randomly generate $2^{n[R_i + R_{i,r}]}$ i.i.d. sequences $v_i^n(m_i, m_{i,r})$, with $(m_i, m_{i,r}) \in [1 : 2^{nR_i}] \times [1 : 2^{nR_{i,r}}]$, each with probability $p(v_i^n | q^n) = \prod_{t=1}^n p(v_i(t) | q(t))$. Every node in the network knows these codebooks. Denote the overall codebook as \mathcal{C} .

Encoding: For $i \in \mathcal{K}$, to send message m_i , transmitter i randomly and uniformly chooses $m_{i,r} \in [1 : 2^{nR_{i,r}}]$ and finds $v_i^n(m_i, m_{i,r})$. Then, given the codeword $v_i^n(m_i, m_{i,r})$, it generates x_i^n according to $\sum_{t=1}^n p(x_i(t) | v_i(t))$ and transmits this sequence to the channel.

Decoding: The legitimate receiver, upon receiving y^n , finds $v_1^n(\hat{m}_1, \hat{m}_{1,r}), v_2^n(\hat{m}_2, \hat{m}_{2,r}), \dots, v_K^n(\hat{m}_K, \hat{m}_{K,r})$ such that $(v_1^n(\hat{m}_1, \hat{m}_{1,r}), v_2^n(\hat{m}_2, \hat{m}_{2,r}), \dots, v_K^n(\hat{m}_K, \hat{m}_{K,r}), y^n)$ is jointly typical.

Analysis of the error probability of decoding: Consider the expected value of the error probability of decoding over the ensemble of random codes \mathcal{C} , i.e., $P_e = \mathbb{E}[P_e(\mathcal{C})]$. Note that here \mathcal{C} denotes the random variable that represents the randomly generated codebook that adhere to the above scheme. From the decoding analysis for the multiple access channel, see, e.g., [13], P_e can be made approximately zero as $n \rightarrow \infty$ if

$$\sum_{j \in \mathcal{J}} [R_j + R_{j,r}] \leq I(V_{\mathcal{J}}; Y | V_{\mathcal{J}^c}, Q), \quad \forall \mathcal{J} \subseteq \mathcal{K}. \quad (7)$$

Analysis of joint secrecy: For the joint secrecy as defined in (3), we show in the following that $\mathbb{E}[R_{L,\mathcal{K}}(\mathcal{C})] \leq \tau_n$. To this end, we show that $H(M_{\mathcal{K}} | Z^n, Q^n, \mathcal{C}) \geq n \sum_{i \in \mathcal{K}} R_i - n\tau_n$, as this implies $I(M_{\mathcal{K}}; Z^n | \mathcal{C}) \leq I(M_{\mathcal{K}}; Z^n, Q^n | \mathcal{C}) \leq n\tau_n$.

$$\begin{aligned} &H(M_{\mathcal{K}} | Z^n, Q^n, \mathcal{C}) = H(M_{\mathcal{K}}, Z^n | Q^n, \mathcal{C}) - H(Z^n | Q^n, \mathcal{C}) \\ &= H(M_{\mathcal{K}}, M_{\mathcal{K},r} | Q^n, \mathcal{C}) + H(Z^n | M_{\mathcal{K}}, M_{\mathcal{K},r}, Q^n, \mathcal{C}) \\ &\quad - H(Z^n | Q^n, \mathcal{C}) - H(M_{\mathcal{K},r} | M_{\mathcal{K}}, Z^n, Q^n, \mathcal{C}) \\ &\stackrel{(a)}{=} H(M_{\mathcal{K}}, M_{\mathcal{K},r} | Q^n, \mathcal{C}) + H(Z^n | V_1^n, \dots, V_K^n, M_{\mathcal{K}}, M_{\mathcal{K},r}, Q^n, \mathcal{C}) \\ &\quad - H(Z^n | Q^n, \mathcal{C}) - H(M_{\mathcal{K},r} | M_{\mathcal{K}}, Z^n, Q^n, \mathcal{C}) \\ &\stackrel{(b)}{=} n \sum_{i \in \mathcal{K}} [R_i + R_{i,r}] + H(Z^n | V_1^n, V_2^n, \dots, V_K^n, Q^n, \mathcal{C}) \end{aligned}$$

$$\begin{aligned}
& -H(Z^n|Q^n, \mathcal{C}) - H(M_{\mathcal{K},r}|M_{\mathcal{K}}, Z^n, Q^n, \mathcal{C}) \\
& \stackrel{(c)}{\geq} n \sum_{i \in \mathcal{K}} [R_i + R_{i,r}] - I(V_1^n, \dots, V_K^n; Z^n|Q^n, \mathcal{C}) \\
& - n \left[\sum_{i \in \mathcal{K}} R_{i,r} - I(V_{\mathcal{K}}; Z|Q) + \varepsilon_n \right] \stackrel{(d)}{\geq} n \sum_{i \in \mathcal{K}} R_i - n\tau_n,
\end{aligned}$$

where (a) follows from the fact that V_1^n, \dots, V_K^n are functions of $(M_1, M_{1,r}), \dots, (M_K, M_{K,r})$, respectively, given Q^n and \mathcal{C} ; (b) follows from the fact that $H(M_{\mathcal{K}}, M_{\mathcal{K},r}|Q^n, \mathcal{C}) = n \sum_{i \in \mathcal{K}} [R_i + R_{i,r}]$, and given Q^n and \mathcal{C} , $(M_{\mathcal{K}}, M_{\mathcal{K},r}) \rightarrow (V_1^n, V_2^n, \dots, V_K^n) \rightarrow Z^n$ forms a Markov chain; (c) follows from Lemma 6 (with $R_{v,i} = R_{i,r}$ for $i \in \mathcal{K}$) by requiring (5):

$$\sum_{j \in \mathcal{J}} R_{j,r} \geq I(V_{\mathcal{J}}; Z|Q), \quad \forall \mathcal{J} \subseteq \mathcal{K}. \quad (8)$$

(d) is due to the fact that $I(V_1^n, \dots, V_K^n; Z^n|Q^n, \mathcal{C}) \leq n[I(V_1, \dots, V_K; Z|Q) + \varepsilon_n]$, the proof of which follows the proof of [14, Lemma 3], and taking $\tau_n = 2\varepsilon_n$.

Joint secrecy achievable rate region: We summarize the requirements in order to guarantee a reliable communication under the joint secrecy constraint as follows:

- the non-negativity for rates;
- the conditions for a reliable communication, i.e., (7);
- the conditions for the joint secrecy, i.e., (8).

That is, we have the following system of inequalities:

$$\begin{aligned}
& \sum_{j \in \mathcal{J}} [R_j + R_{j,r}] \leq I(V_{\mathcal{J}}; Y|V_{\mathcal{J}^c}, Q), \quad \text{for all } \mathcal{J} \subseteq \mathcal{K}, \\
& \sum_{j \in \mathcal{J}} R_{j,r} \geq I(V_{\mathcal{J}}; Z|Q), \quad \text{for all } \mathcal{J} \subseteq \mathcal{K}.
\end{aligned}$$

To obtain the desired region of $\{R_i|i \in \mathcal{K}\}$, the variables of $\{R_{i,r}|i \in \mathcal{K}\}$ are to be eliminated. (The rate constraints $R_i \geq 0$ for $i \in \mathcal{K}$ are not included here, since they will not be involved in the Fourier-Motzkin elimination. But they will be included in the final derived region.)

Fourier-Motzkin Elimination: A matrix notation of the above system of inequalities can be written as follows:

$$\underbrace{\begin{bmatrix} [1_{\mathcal{J}}] \\ [0_{\mathcal{J}}] \end{bmatrix}}_{A'} \cdot \underbrace{\begin{bmatrix} R_1 \\ \vdots \\ R_K \end{bmatrix}}_{x'} + \underbrace{\begin{bmatrix} [1_{\mathcal{J}}] \\ -[1_{\mathcal{J}}] \end{bmatrix}}_{A''} \cdot \underbrace{\begin{bmatrix} R_{1,r} \\ \vdots \\ R_{K,r} \end{bmatrix}}_{x''} \leq \underbrace{\begin{bmatrix} [b_{\mathcal{J}}^+] \\ -[b_{\mathcal{J}}^-] \end{bmatrix}}_b, \quad (9)$$

where

- $1_{\mathcal{J}}$ is the $1 \times K$ indicator vector of the subset \mathcal{J} of the set \mathcal{K} (which definition is given in Definition 1);
- Since $\mathcal{J} = \emptyset$ introduces only redundant inequations, only the non-empty choices of \mathcal{J} need to be considered. Let $\{\mathcal{J}_1, \dots, \mathcal{J}_{2^K-1}\}$ be the list of $2^K - 1$ different non-empty subsets of \mathcal{K} . We have $\mathcal{J} \in \{\mathcal{J}_1, \dots, \mathcal{J}_{2^K-1}\}$;
- $[1_{\mathcal{J}}]$ is a $(2^K - 1) \times K$ matrix, with the i -th row to be the indicator vector $1_{\mathcal{J}_i}$, where $1 \leq i \leq 2^K - 1$. Correspondingly,

- $[b_{\mathcal{J}}^+]$ is a $(2^K - 1) \times 1$ matrix with the i -th row to be $b_{\mathcal{J}_i}^+ = I(V_{\mathcal{J}_i}; Y|V_{\mathcal{J}_i^c}, Q)$,
- $[b_{\mathcal{J}}^-]$ is a $(2^K - 1) \times 1$ matrix with the i -th row to be $b_{\mathcal{J}_i}^- = I(V_{\mathcal{J}_i}; Z|Q)$;
- $[0_{\mathcal{J}}]$ is a $(2^K - 1) \times K$ matrix with zero elements.

Properties of $b_{\mathcal{J}}^+$ and $b_{\mathcal{J}}^-$ are given in Lemma 8 in Appendix A, which play a crucial role in removing the redundant inequations so as to avoid the double exponential complexity of the Fourier-Motzkin elimination.

Note that A', A'' are both $(2^{K+1} - 2) \times K$ matrices. In particular, we have

$$A'' = \begin{bmatrix} [1_{\mathcal{J}}] \\ -[1_{\mathcal{J}}] \end{bmatrix} = B \otimes [1_{\mathcal{J}}], \quad \text{where } B = \begin{bmatrix} 1 \\ -1 \end{bmatrix}. \quad (10)$$

Here \otimes is the Kronecker product. According to [15, Theorem 1], we are looking for a G such that $\{wA'x' \leq wb|w \in G\}$ is equivalent to the final system of the Fourier-Motzkin elimination. Here, G is a base of essentially different minimal vectors of F that is the cone of non-negative solution $w \geq 0$ of $wA'' = 0$.

In fact, for our system defined by (9), we find

$$G = D \otimes I_{2^K-1}, \quad \text{where } D = \begin{bmatrix} 1 & 1 \end{bmatrix}. \quad (11)$$

Here I_n is an $n \times n$ identity matrix. In the following, we show that G (as defined in (11)) is a base of essentially different minimal vectors of F . Note that the system defined by $\{wA'x' \leq wb|w \in G\}$ is:

$$GA'x' \leq Gb \Leftrightarrow [1_{\mathcal{J}}]x' \leq [b_{\mathcal{J}}^+ - b_{\mathcal{J}}^-]. \quad (12)$$

Firstly, it is clear that $w \in G$ is positive, i.e., $w \geq 0$.

Secondly, we show that $wA'' = 0$ for any $w \in G$. Note that $A'' = B \otimes [1_{\mathcal{J}}]$ by (10). Therefore, if $w \cdot (B \otimes [1_{\mathcal{J}}]) = 0$, we have $w \cdot A'' = 0$ as well. It is easy to verify that

$$G \cdot (B \otimes [1_{\mathcal{J}}]) = (D \otimes I_{2^K-1}) \cdot (B \otimes [1_{\mathcal{J}}]) \stackrel{(a)}{=} (DB) \otimes [1_{\mathcal{J}}] = [0_{\mathcal{J}}],$$

where (a) is due to the mixed-product property of the Kronecker product: $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$.

Furthermore, by definition of G in (11), each row of G is essentially different from the others. And, they are all minimal according to [15, Theorem 2].

Most importantly, we show in the following that any other positive vectors c satisfying $c \cdot A'' = 0$, will produce only redundant inequations. For convenience, we denote $c = [c^+ \ c^-]$, with $c^+ = [c_1^+, \dots, c_{2^K-1}^+]$ and $c^- = [c_1^-, \dots, c_{2^K-1}^-]$. Then, $c \cdot A'' = 0$ is equivalent to $c^+ \cdot [1_{\mathcal{J}}] = c^- \cdot [1_{\mathcal{J}}]$.

Recall that for $1 \leq i \leq 2^K - 1$, $1_{\mathcal{J}_i}$ is the indicator vector of \mathcal{J}_i , and thus $c_i^+ > 0$ (or $c_i^- > 0$) indicates c_i^+ (or c_i^-) number of presences of \mathcal{J}_i . Let

$$\begin{aligned}
J_{c^+} &= \underbrace{\{\mathcal{J}_1, \dots, \mathcal{J}_1\}}_{c_1^+}, \dots, \underbrace{\{\mathcal{J}_i, \dots, \mathcal{J}_i\}}_{c_i^+}, \dots, \underbrace{\{\mathcal{J}_{2^K-1}, \dots, \mathcal{J}_{2^K-1}\}}_{c_{2^K-1}^+}; \\
J_{c^-} &= \underbrace{\{\mathcal{J}_1, \dots, \mathcal{J}_1\}}_{c_1^-}, \dots, \underbrace{\{\mathcal{J}_i, \dots, \mathcal{J}_i\}}_{c_i^-}, \dots, \underbrace{\{\mathcal{J}_{2^K-1}, \dots, \mathcal{J}_{2^K-1}\}}_{c_{2^K-1}^-}.
\end{aligned}$$

Denote $n^+ = \sum_{i=1}^{2^K-1} c_i^+$ and $n^- = \sum_{i=1}^{2^K-1} c_i^-$. Simply representing $J_{c^+} = \{\mathcal{J}_1^+, \mathcal{J}_2^+, \dots, \mathcal{J}_{n^+}^+\}$ and $J_{c^-} = \{\mathcal{J}_1^-, \mathcal{J}_2^-, \dots, \mathcal{J}_{n^-}^-\}$, we have

$$c^+ \cdot [1_{\mathcal{J}}] = \sum_{i=1}^{n^+} 1_{\mathcal{J}_i^+}, \quad c^- \cdot [1_{\mathcal{J}}] = \sum_{i=1}^{n^-} 1_{\mathcal{J}_i^-}; \quad (13)$$

$$c^+ \cdot [b_{\mathcal{J}}^+] = \sum_{i=1}^{n^+} b_{\mathcal{J}_i^+}^+, \quad c^- \cdot [b_{\mathcal{J}}^-] = \sum_{i=1}^{n^-} b_{\mathcal{J}_i^-}^-. \quad (14)$$

Note that $c^+ \cdot [1_{\mathcal{J}}] = c^- \cdot [1_{\mathcal{J}}]$ and (13) together imply that both J_{c^+} and J_{c^-} share the same presence vector $c_{\#} = c^+ \cdot [1_{\mathcal{J}}]$ (according to Definition 2). Denote the largest element of $c_{\#}$ to be c_{\max} .

Let $\{\mathcal{J}_i^{\oplus} | 1 \leq i \leq n^+\}$ and $\{\mathcal{J}_i^{\ominus} | 1 \leq i \leq n^-\}$ be the compact forms of the element rearrangement (definition of which is given in Definition 3) for J_{c^+} and J_{c^-} , respectively. Then, according to Lemma 5, they also share the same compact form of the element rearrangement without empty sets, i.e., $\{\mathcal{J}_i^{\oplus} | 1 \leq i \leq c_{\max}\}$. That is,

$$\begin{aligned} \mathcal{J}_i^{\oplus} &= \mathcal{J}_i^{\ominus}, \quad \text{for } 1 \leq i \leq c_{\max}; \\ \mathcal{J}_i^{\oplus} &= \mathcal{J}_j^{\ominus} = \emptyset, \quad \text{for } c_{\max} < i \leq n^+ \text{ \& } c_{\max} < j \leq n^-. \end{aligned} \quad (15)$$

Since the element arrangement does not change the presence vector, we have by Lemma 4:

$$c_{\#} = c^+ \cdot [1_{\mathcal{J}}] = \sum_{i=1}^{c_{\max}} 1_{\mathcal{J}_i^{\oplus}}. \quad (16)$$

Now if we sum up the rows of (12) with respect to $\{\mathcal{J}_i^{\oplus} | 1 \leq i \leq c_{\max}\}$, we obtain the following inequation:

$$\sum_{i=1}^{c_{\max}} 1_{\mathcal{J}_i^{\oplus}} x' \leq \sum_{i=1}^{c_{\max}} [b_{\mathcal{J}_i^{\oplus}}^+ - b_{\mathcal{J}_i^{\oplus}}^-]. \quad (17)$$

On the other hand, if we apply c to the original system (9), we obtain the following inequation:

$$\begin{aligned} cA'x' \leq cb &\Rightarrow c^+ [1_{\mathcal{J}}] x' \leq c^+ [b_{\mathcal{J}}^+] - c^- [b_{\mathcal{J}}^-] \\ &\stackrel{(a)}{\Rightarrow} c_{\#} x' \leq \sum_{i=1}^{n^+} b_{\mathcal{J}_i^+}^+ - \sum_{i=1}^{n^-} b_{\mathcal{J}_i^-}^-, \end{aligned} \quad (18)$$

where (a) is due to (16) and (14).

Note that the LHS of (18) is the same as the LHS of (17) (according to (16)). However, we can show that

$$\begin{aligned} \text{RHS of (18)} &= \sum_{i=1}^{n^+} b_{\mathcal{J}_i^+}^+ - \sum_{i=1}^{n^-} b_{\mathcal{J}_i^-}^- \stackrel{(b)}{\geq} \sum_{i=1}^{n^+} b_{\mathcal{J}_i^{\oplus}}^+ - \sum_{i=1}^{n^-} b_{\mathcal{J}_i^{\ominus}}^- \\ &\stackrel{(c)}{=} \sum_{i=1}^{c_{\max}} [b_{\mathcal{J}_i^{\oplus}}^+ - b_{\mathcal{J}_i^{\oplus}}^-] = \text{RHS of (17)}, \end{aligned}$$

where (b) is according to Lemma 8 (as given in Appendix A) and (c) is due to (15) and $b_{\emptyset}^+ = b_{\emptyset}^- = 0$. That is, (18) is redundant since it is already implied by (17), which can be derived as a linear combinatory with positive coefficients of inequations of (12). Since this applies to any positive

vector c that is other than the rows of G (as defined in (11)) such that $c \cdot A'' = 0$, therefore, the final system of (9) is equivalent to the one (12).

As a conclusion, (12) establishes the resulting joint secrecy region. ■

APPENDIX A PROPERTIES OF $b_{\mathcal{T}}^+$ AND $b_{\mathcal{T}}^-$

Recall that $b_{\mathcal{T}}^+ = I(V_{\mathcal{T}}; Y | V_{\mathcal{T}^c}, Q)$ and $b_{\mathcal{T}}^- = I(V_{\mathcal{T}}; Z | Q)$ as defined in (9). We have the following lemma:

Lemma 8. *Given $\{\mathcal{T}_i | 1 \leq i \leq t\}$ as a list of t subsets of \mathcal{K} , and its compact form of the element rearrangement $\{\mathcal{T}_{t,i}^* | 1 \leq i \leq t\}$, we have*

$$\sum_{i=1}^t b_{\mathcal{T}_i}^+ \geq \sum_{i=1}^t b_{\mathcal{T}_{t,i}^*}^+ \quad \text{and} \quad \sum_{i=1}^t b_{\mathcal{T}_i}^- \leq \sum_{i=1}^t b_{\mathcal{T}_{t,i}^*}^-.$$

To prove Lemma 8, we need the following two lemmas.

Lemma 9. *Given $\{\mathcal{T}_i | 1 \leq i \leq t-1\}$ as a list of $t-1$ subsets of \mathcal{K} , and list $\{\mathcal{T}_i | 1 \leq i \leq t\}$ (with one more subset \mathcal{T}_t included), let $\{\mathcal{T}_{t-1,i}^* | 1 \leq i \leq t-1\}$ and $\{\mathcal{T}_{t,i}^* | 1 \leq i \leq t\}$ be their compact forms of the element rearrangement, respectively. We have*

$$\mathcal{T}_{t,i}^* = \begin{cases} \mathcal{T}_{t-1,1}^* \cup \mathcal{T}_t & i = 1 \\ \mathcal{T}_{t-1,i}^* \cup (\mathcal{T}_{t-1,i-1}^* \cap \mathcal{T}_t) & 1 < i < t \\ \mathcal{T}_{t-1,t-1}^* \cap \mathcal{T}_t & i = t \end{cases}.$$

Proof: The proof for $i = 1, t$ is straightforward by definitions of $\mathcal{T}_{t,1}^*$ and $\mathcal{T}_{t,t}^*$. For $1 < i < t$, the proof is given in steps (19)-(21) (at the top of the next page), where (19) is by the definition of $\mathcal{T}_{t,i}^*$; (20) is due to the distributive laws of the sets and (21) is by the definition of $\mathcal{T}_{t-1,i}^*$. ■

Lemma 10. *For $\forall \mathcal{T}_1, \mathcal{T}_2 \subseteq \mathcal{K}$, we have*

- 1) $b_{\mathcal{T}_1}^+ + b_{\mathcal{T}_2}^+ \geq b_{\mathcal{T}_1 \cap \mathcal{T}_2}^+ + b_{\mathcal{T}_1 \cup \mathcal{T}_2}^+$;
- 2) $b_{\mathcal{T}_1}^- + b_{\mathcal{T}_2}^- \leq b_{\mathcal{T}_1 \cap \mathcal{T}_2}^- + b_{\mathcal{T}_1 \cup \mathcal{T}_2}^-$.

Proof: We only give the proof for $b_{\mathcal{T}_1}^+ + b_{\mathcal{T}_2}^+$. A similar proof applies to $b_{\mathcal{T}_1}^- + b_{\mathcal{T}_2}^-$.

$$\begin{aligned} b_{\mathcal{T}_1}^+ + b_{\mathcal{T}_2}^+ &\stackrel{(a)}{=} I(V_{\mathcal{T}_1}; Y | V_{\mathcal{T}_1^c}, Q) + I(V_{\mathcal{T}_2}; Y | V_{\mathcal{T}_2^c}, Q) \\ &\stackrel{(b)}{=} I(V_{\mathcal{T}_1 \cap \mathcal{T}_2}, V_{\mathcal{T}_1 \cap \mathcal{T}_2^c}; Y | V_{\mathcal{T}_1^c}, Q) + I(V_{\mathcal{T}_2}; Y | V_{\mathcal{T}_2^c}, Q) \\ &\stackrel{(c)}{=} I(V_{\mathcal{T}_1 \cap \mathcal{T}_2^c}; Y | V_{\mathcal{T}_1^c}, Q) + I(V_{\mathcal{T}_1 \cap \mathcal{T}_2}; Y | V_{(\mathcal{T}_1 \cap \mathcal{T}_2)^c}, Q) \\ &\quad + I(V_{\mathcal{T}_2}; Y | V_{\mathcal{T}_2^c}, Q) \\ &\stackrel{(d)}{\geq} I(V_{\mathcal{T}_1 \cap \mathcal{T}_2}; Y | V_{(\mathcal{T}_1 \cap \mathcal{T}_2)^c}, Q) + I(V_{\mathcal{T}_1 \cap \mathcal{T}_2^c}; Y | V_{\mathcal{T}_1^c \cap \mathcal{T}_2^c}, Q) \\ &\quad + I(V_{\mathcal{T}_2}; Y | V_{\mathcal{T}_1 \cap \mathcal{T}_2^c}, V_{\mathcal{T}_1^c \cap \mathcal{T}_2^c}, Q) \\ &\stackrel{(e)}{=} I(V_{\mathcal{T}_1 \cap \mathcal{T}_2}; Y | V_{(\mathcal{T}_1 \cap \mathcal{T}_2)^c}, Q) + I(V_{\mathcal{T}_1 \cup \mathcal{T}_2}; Y | V_{(\mathcal{T}_1 \cup \mathcal{T}_2)^c}, Q), \end{aligned}$$

where (a) is by the definition of $b_{\mathcal{T}}^+$; (b) is by the fact that $\mathcal{T}_1 = (\mathcal{T}_1 \cap \mathcal{T}_2) \cup (\mathcal{T}_1 \cap \mathcal{T}_2^c)$; (c) is by the chain rule of the mutual information; (d) is by the facts that $\mathcal{T}_2^c = (\mathcal{T}_1 \cap \mathcal{T}_2^c) \cup (\mathcal{T}_1^c \cap \mathcal{T}_2^c)$ and $I(V_{\mathcal{T}_1 \cap \mathcal{T}_2^c}; Y | V_{\mathcal{T}_1^c}, Q) = I(V_{\mathcal{T}_1 \cap \mathcal{T}_2^c}; V_{\mathcal{T}_1^c \cap \mathcal{T}_2^c}, Y | V_{\mathcal{T}_1^c \cap \mathcal{T}_2^c}, Q) \geq I(V_{\mathcal{T}_1 \cap \mathcal{T}_2}; Y | V_{\mathcal{T}_1^c \cap \mathcal{T}_2^c}, Q)$.

$$\mathcal{T}_{t,i}^* = \bigcup_{\{j_1, \dots, j_i\} \subseteq [1:t]} \left(\bigcap_{k=1}^i \mathcal{T}_{j_k} \right) = \left\{ \bigcup_{\{j_1, \dots, j_i\} \subseteq [1:t-1]} \left(\bigcap_{k=1}^i \mathcal{T}_{j_k} \right) \right\} \cup \left\{ \bigcup_{\{j_1, \dots, j_{i-1}\} \subseteq [1:t-1]} \left(\left(\bigcap_{k=1}^{i-1} \mathcal{T}_{j_k} \right) \cap \mathcal{T}_t \right) \right\} \quad (19)$$

$$= \left\{ \bigcup_{\{j_1, \dots, j_i\} \subseteq [1:t-1]} \left(\bigcap_{k=1}^i \mathcal{T}_{j_k} \right) \right\} \cup \left\{ \left(\bigcup_{\{j_1, \dots, j_{i-1}\} \subseteq [1:t-1]} \left(\bigcap_{k=1}^{i-1} \mathcal{T}_{j_k} \right) \right) \cap \mathcal{T}_t \right\} \quad (20)$$

$$= \mathcal{T}_{t-1,i}^* \cup \left(\mathcal{T}_{t-1,i-1}^* \cap \mathcal{T}_t \right), \quad (21)$$

Q), which holds since $V_{\mathcal{T}_1 \cap \mathcal{T}_2^c}$, $V_{\mathcal{T}_1^c \cap \mathcal{T}_2}$ and $V_{\mathcal{T}_1^c \cap \mathcal{T}_2^c}$ are independent given Q ; and (e) is due to the facts that $\mathcal{T}_1 \cup \mathcal{T}_2 = \mathcal{T}_2 \cup (\mathcal{T}_1 \cap \mathcal{T}_2^c)$ and $\mathcal{T}_1^c \cap \mathcal{T}_2^c = (\mathcal{T}_1 \cup \mathcal{T}_2)^c$. ■

Now we give the proof of Lemma 8 as follows.

First, we show that the statement $\sum_{i=1}^t b_{\mathcal{T}_i}^+ \geq \sum_{i=1}^t b_{\mathcal{T}_{t,i}^*}^+$ is true by induction.

- For $t = 1$, we have by definition $\mathcal{T}_{1,1}^* = \mathcal{T}_1$. Thus, $b_{\mathcal{T}_1}^+ = b_{\mathcal{T}_{1,1}^*}^+$ and the statement is true for $t = 1$.
- For $t = 2$, the statement is true by Lemma 10.
- Assume that the statement is true for some natural number $t - 1$. That is, for any $\{\mathcal{T}_i | 1 \leq i \leq t - 1\}$ as a list of $t - 1$ subsets of \mathcal{K} , and its compact form of element rearrangement $\{\mathcal{T}_{t-1,i}^* | 1 \leq i \leq t - 1\}$, we have $\sum_{i=1}^{t-1} b_{\mathcal{T}_i}^+ \geq \sum_{i=1}^{t-1} b_{\mathcal{T}_{t-1,i}^*}^+$.
- Now we show that the statement is also true for t .

$$\begin{aligned} \sum_{i=1}^t b_{\mathcal{T}_i}^+ &= \sum_{i=1}^{t-1} b_{\mathcal{T}_i}^+ + b_{\mathcal{T}_t}^+ \stackrel{(a)}{\geq} \sum_{i=1}^{t-1} b_{\mathcal{T}_{t-1,i}^*}^+ + b_{\mathcal{T}_t}^+ \\ &= \sum_{i=2}^{t-1} b_{\mathcal{T}_{t-1,i}^*}^+ + \left(b_{\mathcal{T}_{t-1,1}^*}^+ + b_{\mathcal{T}_t}^+ \right) \\ &\stackrel{(b_1)}{\geq} \sum_{i=2}^{t-1} b_{\mathcal{T}_{t-1,i}^*}^+ + \left(b_{\mathcal{T}_{t-1,1}^* \cap \mathcal{T}_t}^+ + b_{\mathcal{T}_{t-1,1}^* \cup \mathcal{T}_t}^+ \right) \\ &\stackrel{(c_1)}{=} \sum_{i=3}^{t-1} b_{\mathcal{T}_{t-1,i}^*}^+ + \left(b_{\mathcal{T}_{t-1,2}^*}^+ + b_{\mathcal{T}_{t-1,1}^* \cap \mathcal{T}_t}^+ \right) + b_{\mathcal{T}_{t,1}^*}^+ \\ &\stackrel{(b_2)}{\geq} \sum_{i=3}^{t-1} b_{\mathcal{T}_{t-1,i}^*}^+ + \left(b_{\mathcal{T}_{t-1,2}^* \cap \mathcal{T}_t}^+ + b_{\mathcal{T}_{t-1,2}^* \cup (\mathcal{T}_{t-1,1}^* \cap \mathcal{T}_t)}^+ \right) \\ &\quad + b_{\mathcal{T}_{t,1}^*}^+ \\ &\stackrel{(c_2)}{=} \sum_{i=4}^{t-1} b_{\mathcal{T}_{t-1,i}^*}^+ + \left(b_{\mathcal{T}_{t-1,3}^*}^+ + b_{\mathcal{T}_{t-1,2}^* \cap \mathcal{T}_t}^+ \right) + \sum_{i=1}^2 b_{\mathcal{T}_{t,i}^*}^+ \\ &\quad \vdots \\ &\stackrel{(b_{t-1})}{\geq} b_{\mathcal{T}_{t-1,t-1}^* \cap \mathcal{T}_t}^+ + b_{\mathcal{T}_{t-1,t-1}^* \cup (\mathcal{T}_{t-1,t-2}^* \cap \mathcal{T}_t)}^+ + \sum_{i=1}^{t-2} b_{\mathcal{T}_{t,i}^*}^+ \\ &\stackrel{(c_{t-1})}{=} \sum_{i=1}^t b_{\mathcal{T}_{t,i}^*}^+, \end{aligned}$$

where (a) is due to the fact that the statement is true for $t - 1$; and for $1 \leq j \leq t - 1$, step (b_j) is by applying

Lemma 10 and the fact that $\mathcal{T}_{t-1,j}^* \cap (\mathcal{T}_{t-1,j-1}^* \cap \mathcal{T}_t) = \mathcal{T}_{t-1,j}^* \cap \mathcal{T}_t$ for $2 \leq j \leq t - 1$ (since $\mathcal{T}_{t-1,j}^* \subseteq \mathcal{T}_{t-1,j-1}^*$ by definition); step (c_j) is by applying Lemma 9. In particular, (c₁) is by the fact that $\mathcal{T}_{t,1}^* = \mathcal{T}_{t-1,1}^* \cup \mathcal{T}_t$; step (c_{t-1}) is by the fact that $\mathcal{T}_{t,t}^* = \mathcal{T}_{t-1,t-1}^* \cap \mathcal{T}_t$; and other intermediate steps are by the fact that $\mathcal{T}_{t-1,j}^* \cup (\mathcal{T}_{t-1,j-1}^* \cap \mathcal{T}_t) = \mathcal{T}_{t,j}^*$ for $1 < j < t$.

A similar proof can be applied to show that the statement for $\sum_{i=1}^t b_{\mathcal{T}_i}^-$ is also true. This concludes our proof.

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